Numerical Evolution of Puncture Black Holes

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Here, I summarize the current status of the puncture black hole project. The first problem to solve is distorted black holes. The second problem to solve is head-on collision of (non-)equal mass binary puncture with mesh refinement.

PACS numbers:

INTRODUCTION

INITIAL DATA

Multi-Blackholes

For a generic puncture black hole initial data looks like

$$g_{ij} \; = \; \psi^4 \tilde{g_{ij}} \tag{1}$$

where

$$\psi = 1 + \sum_{n=1}^{N_p} \frac{m_n}{2|\vec{r} - \vec{c}_n|} + u \tag{2}$$

Assuming no linear momentum and no spin, $u = 0$. Otherwise, Hamiltonian Constraint Equation needs to be solved.
Now in the `Handol` code,

$$\psi = e^{\phi} \tag{3}$$
$$\phi = \xi + \ln \psi_{BL} \tag{4}$$
$$\tag{5}$$

where ξ is a regular part and $ln\psi_{LB}$ a singular part which we will treat analytically.

$$\psi_{BL} = 1 + \sum_{n=1}^{N_p} \frac{m_n}{2|\vec{r} - \vec{c}_n|} \tag{6}$$

Therefore,

$$\ln \psi_{BL} \; = \; \ln(1 + \sum_{n=1}^{N_p} \frac{m_n}{2|\vec{r} - \vec{c}_n|}) \tag{7}$$

$$\partial_i \ln \psi_{BL} \; = \; \frac{1}{\psi_{BL}} \partial_i \psi_{BL} = \frac{1}{\psi_{BL}} \frac{-1}{2} \sum_{n=1}^{N_p} m_n \frac{r_i - c_{n,i}}{|\vec{r} - \vec{c}_n|^3} \tag{8}$$

$$\begin{aligned}\partial_i \partial_j \ln \psi_{BL} \; &= \; \frac{-1}{2} \{ -\frac{1}{\psi_{BL}^2} (\partial_j \psi_{BL}) \sum_{n=1}^{N_p} \frac{m_n (r_i - c_{n,i})}{|\vec{r} - \vec{c}_n|^3} \\ &+ \frac{1}{\psi_{BL}} \sum_{n=1}^{N_p} m_n \partial_j \frac{(r_i - c_{n,i})}{|\vec{r} - \vec{c}_n|^3} \}\end{aligned} \tag{9}$$

$$\partial_j \frac{(r_i - c_{n,i})}{|\vec{r} - \vec{c}_n|^3} \; = \; \frac{\delta_{ij} |\vec{r} - \vec{c}_n|^3 - 3(r_i - c_{n,i})(r_j - c_{n,j}) |\vec{r} - \vec{c}_n|}{|\vec{r} - \vec{c}_n|^6} \tag{10}$$

$$\partial_i\partial_j \ln\psi_{BL} = \frac{-1}{2}\{-\frac{1}{\psi_{BL}^2}(\frac{-1}{2}\sum_{n=1}^{N_p}\frac{m_n(r_j-c_{n,j})}{|\vec{r}-\vec{c}_n|^3})\sum_{n=1}^{N_p}\frac{m_n(r_i-c_{n,i})}{|\vec{r}-\vec{c}_n|^3} + \frac{1}{\psi_{BL}}\sum_{n=1}^{N_p} m_n \frac{\delta_{ij}|\vec{r}-\vec{c}_n|^3 - 3(r_i-c_{n,i})(r_j-c_{n,j})|\vec{r}-\vec{c}_n|}{|\vec{r}-\vec{c}_n|^6}\} \quad (11)$$

Distorted Blackhole

Here I will try to describe some of my ideas about how to set up distorted blackhole initial data using puncture method.

In 3+1 approach, constraint equations to solve are given by,

$$R + K^2 - K^{ij}K_{ij} = 0 \quad (12)$$
$$D_i(K^{ij} - \gamma^{ij}K) = 0 \quad (13)$$

Applying York's conformal decomposition,

$$\gamma_{ij} = \psi^4\tilde{\gamma_{ij}} \quad (14)$$
$$K_{ij} = \psi^{-2}\tilde{K_{ij}} \quad (15)$$

For now, I won't concern momentum constraint equation (MCE) and assume somehow we will find a way solve either analytically and numerically MCE. Ok, this may not be an easy task in general, will be back later. (Also if we assume maximal slicing, $K = 0$, things simplify.)

Hamiltonian constraint equation (HCE) becomes,

$$\tilde{\Delta}\psi = \frac{1}{8}\psi\tilde{R} - \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{12}\psi^5K^2 \quad (16)$$

Now employing puncture idea of seperating singular part,

$$\psi = \frac{1}{\alpha} + u \quad (17)$$

where

$$\frac{1}{\alpha} = \sum_{i=1}^{N}\frac{m_i}{2|\vec{r}-\vec{r_i}|} \quad (18)$$

(In the following, I will only consider a simple case of single distorted blackhole, $N = 1, \vec{r_i} = \vec{0}$.)

Then, HCE becomes

$$\tilde{\Delta}u = \frac{1}{8}\psi\tilde{R} - \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{12}\psi^5K^2 \quad (19)$$

For the puncture method to work, RHS should be everywhere finite. Now, if near the puncture, $\tilde{A}_{ij}$ diverges no faster than $1/r^3$ and K vanishes at least as fast as r^3, then 2nd and 3rd terms are ok.

Now let's focus on the 1st term, $\psi\tilde{R}$. Assume the perturbation information is encoded thru Brill-form, i.e., conformal metric is non-flat and has the following form,

$$\tilde{\gamma}_{ij} = e^{2q}(d\rho^2 + dz^2) + \rho d\phi^2 \quad (20)$$

Then, we get,

$$\frac{1}{8}\tilde{R} = -\frac{1}{4}(\frac{\partial^2 q}{\partial\rho^2} + \frac{\partial^2 q}{\partial z^2}) \quad (21)$$

Now let's take a common form of function q that people use,

$$q = A\rho^2 e^{-\frac{\rho^2}{\lambda_\rho^2}-\frac{z^2}{\lambda_z^2}}(1 + c\cos^2\phi) \quad (22)$$

(For now, I will assume $c = 0$, but will be back to the non-zero case of c, non-axisymmetry.)
If we use eqn (22), we get,

$$\frac{1}{8}\tilde{R} \sim Ae^{-\frac{\rho^2}{\lambda_\rho^2}-\frac{z^2}{\lambda_z^2}}(1 + O(\rho^2) + O(\rho^4)) \tag{23}$$

Therefore, if we consider the term in HCE, $\frac{1}{8}\psi\tilde{R}$, when approacing puncture (in this case at the origin), we get a term that diverges!

Now I propose a solution to this problem. Note that we do not have to take the form (22). All we need a some form of q that satisfies regularity conditions with a right asymtotic behavior,

$$q|_{\rho\to} \to 0 \tag{24}$$
$$q_{,\rho}|_{\rho\to} \to 0 \tag{25}$$
$$\tag{26}$$

q is at least $\sim O(r^{-2})$ when approaching asymptotics.

I propose to use the following form (with $c = 0$ for now),

$$q = A\rho^4 e^{-\frac{\rho^2}{\lambda_\rho^2}-\frac{z^2}{\lambda_z^2}}(1 + c\cos^2\phi) \tag{27}$$

then, $\psi\tilde{R}$ remain finite everywhere and regular.

$$\frac{1}{8}\tilde{R} = -\frac{1}{4}q(\frac{12}{\rho^2} - \frac{18}{\lambda_\rho^2} + \frac{4\rho^2}{\lambda_\rho^4} - \frac{2}{\lambda_z^2} + \frac{4z^2}{\lambda_z^4}) \tag{28}$$

Here are now some problems that can be studied with `AMRMG3D` code.

- Time symmetric, non-rotating, axisymmetric distorted blackhole: Use perturbation of the form (27). Note this doesn't have a direct astrophysical implication, but still can be a good exercise to test my idea.
- Time symmetric, non-rotating, non-axisymmetric distorted blackhole: Note, in this case Laplacian operator, $\tilde{\Delta}$, won't be flat Laplcian. And regularity of RHS should be checked again...
- Non time-symmetric, rotating, axisymmetric distorted blackhole: This is more astrophysically relevant because it may approximately represent the spacetime right after the binary blackhole merger. Here probably one is better to use "flat" conformal metric and encode perturbation thru extrinsic curvature. If we want to keep brill form for the conformal metric, then we need to find out the new solutions for extrinsic curvature...
- Non time-symmetric, rotating, non-axisymmetric distorted blackhole: To be further explorered.

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[1] S. Brandt, K. Camarda, E. Seidel, and R. Takahashi, gr-qc/0206070 (2002).
[2] S. Brandt, and B. Brugmann, PRL, 78, 3606 (1997)